

Superconformal Yang-Mills quantum mechanics and Calogero model with $\mathbf{OSp}(\mathcal{N}|2, \mathbb{R})$ symmetry

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Abstract

In spacetime dimension two, pure Yang-Mills possesses no physical degrees of freedom, and consequently it admits a supersymmetric extension to couple to an arbitrary number, \mathcal{N} say, of Majorana-Weyl gauginos. This results in $(\mathcal{N}, 0)$ super Yang-Mills. Further, its dimensional reduction to mechanics doubles the number of supersymmetries, from \mathcal{N} to $\mathcal{N} + \mathcal{N}$, to include conformal supercharges, and leads to a superconformal Yang-Mills quantum mechanics with symmetry group $\mathbf{OSp}(\mathcal{N}|2, \mathbb{R})$. We comment on its connection to $AdS_2 \times S^{\mathcal{N}-1}$ and reduction to a supersymmetric Calogero model.

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1 Introduction and summary

The AdS/CFT correspondence is a holographic duality between string theory in higher dimensional anti-de Sitter space and gauge theory in lower dimensional flat spacetime, the prototypical example being four-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory dual to string theory on an $AdS_5 \times S^5$ background [1]. While other examples in various spacetime dimensions have been much studied, the lowest dimensional case, *i.e.* the AdS_2/CFT_1 correspondence, is least understood [2]. Although there has been much work on conformal and superconformal mechanics [3–10], the connection to string theory and supergravity is less clear: we do not have the now familiar picture of the gravity side as the near-horizon geometry of coincident branes with the

worldvolume gauge theory living on the boundary of the AdS near-horizon region.

Conformal mechanics is not usually formulated as a gauge theory, but as some multi-particle or supersymmetric extension of the original de Alfaro, Fubini and Furlan (DFF) conformal mechanics with inverse x^2 potential [11] given by

$$S = \int dt \left(\dot{x}^2 - \frac{g}{x^2} \right), \quad (1.1)$$

with g constant. This model is related to the Calogero model of a class of integrable multi-particle systems [12, 13] (see also [14, 15]), being obtainable from the two-particle case. Generalizations of these models to higher number of particles, K , and supersymmetry were reviewed recently by Fedoruk, Ivanov and Lechtenfeld [9] (see also [16]). Although these models in their final form do not contain gauge fields, there are a class of supersymmetric models that have been found by gauging models with auxiliary fields in the fundamental representation of a gauge group [9, 17] (some early bosonic work used a similar approach [18]).

In much of the discussion of the AdS_2/CFT_1 correspondence the bulk theory has been considered from a genuine two-dimensional viewpoint (*e.g.* [19–22]). However, it is also possible to embed an AdS_2 into critical string theory by considering an extra factor such as a sphere. Indeed, D0-particles can give rise to $AdS_2 \times S^8$ geometry (*e.g.* see discussion in [23]). Such geometries with a sphere factor commonly appear in the near-horizon geometry of black holes, as for every known extremal black hole the near-horizon geometry contains an AdS_2 factor¹. Compactifying string theory with intersecting D-branes wrapping an internal space gives rise to such black holes, which should be dual to some CFT_1 on the uncompactified time direction of the branes’ world-volume intersection. This has been used from early investigations into the relations with string theory [24] to recent investigations into black hole entropy [25, 26].

A more explicit realization by Claus *et al.* [27] noted that a worldline action of a superparticle in the $AdS_2 \times S^2$ near horizon geometry of an extreme Reissner-Nordström black hole reduced to superconformal mechanics in a certain limit (see also [28, 29]). This allowed Gibbons and

¹ AdS factors can also be seen in black hole moduli spaces [3, 30].

Townsend [23] to argue the CFT_1 describing the brane construction of this black hole was an $\mathcal{N} = 4$ Calogero model, and further that it should be obtainable from a dimensional reduction of the super Yang-Mills on the two-dimensional intersection of two of the branes (the connection between Calogero models and two-dimensional Yang-Mills having been made before [31]).

Pure Yang-Mills theories in spacetime dimensions three, four, six and ten admit a minimal supersymmetric extension. That is to say, without introducing additional bosonic scalar fields, it is possible to match the bosonic and fermionic physical degrees of freedom,

$$\begin{aligned}
D = 3 \quad (\text{Majorana spinor}) : \quad & 3 - 2 = \frac{1}{2} \times 2^1, \\
D = 4 \quad (\text{Majorana spinor}) : \quad & 4 - 2 = \frac{1}{2} \times 2^2, \\
D = 6 \quad (\text{complex Weyl spinor}) : \quad & 6 - 2 = \frac{1}{2} \times (\frac{1}{2} \times 2^3 + \frac{1}{2} \times 2^3), \\
D = 10 \quad (\text{Majorana-Weyl spinor}) : \quad & 10 - 2 = \frac{1}{2} \times \frac{1}{2} \times 2^5.
\end{aligned} \tag{1.2}$$

Technically at the Lagrangian level, the supersymmetry is realized by virtue of Fierz identities that cancel the cubic order terms in gauginos arising from the supersymmetric variation of the Yukawa term. Further, the supersymmetry in the above dimensions also has deep connection to the division algebras [32–35].

On the other hand, in two-dimensional spacetime pure Yang-Mills has no physical degrees of freedom. This hints at the possibility of a supersymmetric extension coupling to an arbitrary number, \mathcal{N} , of gauginos. The matching of the physical degrees will then be

$$2 - 2 = \mathcal{N} \times 0. \tag{1.3}$$

The present work is based on the observation that indeed such a supersymmetric extension is possible. The main contents as well as the organization of this paper are as follows:

- In section 2 we explicitly construct $2D$ $(\mathcal{N}, 0)$ super Yang-Mills action, of which the bosonic sector is simply pure Yang-Mills and the fermionic sector consists of an arbitrary number, \mathcal{N} , of Majorana-Weyl gauginos.

- In section 3.1, we perform a dimensional reduction of $2D$ $(\mathcal{N}, 0)$ super Yang-Mills to a Yang-Mills quantum mechanics. We show that the resulting one-dimensional Yang-Mills is superconformal as the supersymmetry becomes doubled, $\mathcal{N} \rightarrow \mathcal{N} + \mathcal{N}$, to include conformal supercharges.
- In section 3.2, we identify the superconformal symmetry group as $\mathbf{OSp}(\mathcal{N}|2, \mathbb{R})$.
- Further, in section 3.3, we generalize the Yang-Mills quantum mechanics to include an arbitrary time dependent mass term and a one-dimensional Chern-Simons term, without breaking the superconformal symmetry. In the case when the mass parameter is constant, the massive model corresponds to the radial quantization of the massless model.
- In section 4, we consider two different gauge choices, one a complete gauge fixing and the other a partial gauge fixing. In the latter case, we break the gauge group $U(K)$ to $U(1)^K$ by diagonalizing the unique bosonic dynamical matrix and eliminate the off-diagonal components of the gauge field using its equation of motion. This results in a Calogero-like model with inverse square potential. We show that the resultant model maintains $\mathcal{N} + \mathcal{N}$ superconformal symmetries.
- In section 5, we map the geodesic motion of a point particle on $AdS_2 \times S^{\mathcal{N}-1}$ to the Abelian sector of our massive super Yang-Mills quantum mechanics, where the mass is given by the inverse of the AdS_2 radius.
- Section 6 contains concluding remarks.

The enhancement of ordinary supersymmetry to superconformal symmetry upon dimensional reduction is well known for $4D$ super Yang-Mills:² Dimensional reduction of $10D$ minimal super Yang-Mills to $4D$ $\mathcal{N} = 4$ super Yang-Mills doubles the number of supersymmetries from sixteen to thirty two, resulting in the superconformal symmetry, $SU(2, 2|\mathcal{N})$.

²For other examples of the supersymmetry enhancement upon dimensional reduction, see [39, 40].

For earlier studies on the superconformal mechanics and its super Lie algebra, we refer to reviews [3, 8–10, 36–38] and references therein. In particular, in the $\mathcal{N} = 1, 2, 4$ cases our model essentially coincides with [9, 17]. Compared to them, the novelties of our model are that *i*) it is of Yang-Mills type with gauge group, $\mathbf{U}(K)$, *ii*) it has an arbitrary number of supersymmetries, \mathcal{N} , *iii*) it admits an arbitrary time dependent mass deformation as well as a Chern-Simons term, and *iv*) the corresponding superconformal group is $\mathbf{OSp}(\mathcal{N}|2, \mathbb{R})$.

2 $2D(\mathcal{N}, 0)$ super Yang-Mills

The Minkowskian $2D(\mathcal{N}, 0)$ super Yang-Mills Lagrangian we propose is

$$\mathcal{L}_{2D\text{SYM}} = \text{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \frac{1}{2} \bar{\Psi}_a \Gamma^\mu D_\mu \Psi^a \right]. \quad (2.1)$$

Specifically, we assume the gauge group, $\mathbf{U}(K)$, and set in a standard manner,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu], \quad D_\mu \Psi^a = \partial_\mu \Psi^a - i [A_\mu, \Psi^a]. \quad (2.2)$$

The spinors, Ψ^a , $a = 1, 2, \dots, \mathcal{N}$, are Majorana-Weyl of definite chirality,

$$\Gamma^{01} \Psi^a = +\Psi^a, \quad \bar{\Psi}_a = (\Psi^a)^\dagger \Gamma^0 = -\bar{\Psi}_a \Gamma^{01}, \quad (2.3)$$

such that each Ψ^a has only one Hermitian spinorial component,

$$\Psi^a = \begin{pmatrix} 2^{-\frac{1}{4}} \psi^a \\ 0 \end{pmatrix}, \quad \psi^a = (\psi^a)^\dagger. \quad (2.4)$$

Note that both A_μ and ψ^a are $K \times K$ Hermitian matrices in the adjoint representation of the gauge group, $\mathbf{U}(K)$.

The $(\mathcal{N}, 0)$ supersymmetry transformation is given by

$$\delta A_\mu = i\bar{\mathcal{E}}_a \Gamma_\mu \Psi^a = -i\bar{\Psi}_a \Gamma_\mu \mathcal{E}^a, \quad \delta \Psi^a = -\frac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \mathcal{E}^a, \quad (2.5)$$

where \mathcal{E}^a , $a = 1, 2, \dots, \mathcal{N}$ are Majorana-Weyl spinorial Grassmann parameters.

The Lagrangian is invariant up to total derivatives under the supersymmetry transformation. This can be easily seen by employing the light-cone coordinates,

$$x^+ = \frac{1}{\sqrt{2}}(t + x), \quad x^- = \frac{1}{\sqrt{2}}(t - x), \quad \partial_+ = \frac{1}{\sqrt{2}}(\partial_t + \partial_x), \quad \partial_- = \frac{1}{\sqrt{2}}(\partial_t - \partial_x). \quad (2.6)$$

Setting the metric to be

$$ds^2 = -dt^2 + dx^2 = -2dx^+ dx^-, \quad (2.7)$$

we choose the gamma matrices as

$$\Gamma^+ = -\Gamma_- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad \Gamma^- = -\Gamma_+ = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}. \quad (2.8)$$

The chiral spinors then satisfy

$$\Gamma_- \Psi^a = 0, \quad \Gamma_- \mathcal{E}^a = 0, \quad \Gamma^0 \Gamma^\mu D_\mu \Psi = -\sqrt{2} D_- \Psi, \quad (2.9)$$

which along with (2.4) reduces the Lagrangian (2.1) to

$$\mathcal{L}_{2D\text{SYM}} = \text{Tr} \left[\frac{1}{2} (F_{+-})^2 + i \frac{1}{2} \psi_a D_- \psi^a \right]. \quad (2.10)$$

Hereafter $\mathbf{O}(\mathcal{N})$ indices will be contracted with the Kronecker-delta symbols, δ^{ab} , δ_{ab} , using Einstein convention.

The crucial observation is that only A_- couples to the gauginos, while its supersymmetry variation is trivial due to the chirality of spinors (2.9),

$$\delta A_- = i\bar{\mathcal{E}}_a \Gamma_- \Psi^a = 0. \quad (2.11)$$

Thus, unlike higher dimensional super Yang-Mills, no cubic-order terms in gauginos appear from the supersymmetry variation of the Yukawa term. Accordingly, there is no call for any Fierz identity for the arbitrary $(\mathcal{N}, 0)$ supersymmetry to hold!

3 Superconformal Yang-Mills quantum mechanics

In this section, we first perform a dimensional reduction of the above $2D$ $(\mathcal{N}, 0)$ super Yang-Mills to a $1D$ matrix model (Yang-Mills quantum mechanics) and discuss its supersymmetric deformation. We identify its superconformal symmetry group as $\text{OSp}(\mathcal{N}|2, \mathbb{R})$.

3.1 Dimensional reduction

The dimensional reduction of the above $2D$ $(\mathcal{N}, 0)$ super Yang-Mills (2.10) to the $1D$ light-cone time, $x^- \equiv t$, leads to the following super Yang-Mills quantum mechanics (SYMQM),

$$\mathcal{L}_{\text{SYMQM}} = \text{Tr} \left[\frac{1}{2} (D_t X)^2 + i \frac{1}{2} \psi_a D_t \psi^a \right], \quad (3.1)$$

where the covariant derivative takes the form,

$$D_t = \partial_t - i [A, \quad]. \quad (3.2)$$

Note that X is the only bosonic physical variable, the gauge field A is auxiliary, and the fermions ψ^a , $a = 1, 2, \dots, \mathcal{N}$ carry no spinorial index. All the variables, X , A , ψ^a are $K \times K$ Hermitian matrices.

The gauge symmetry is given by, with $g \in \text{U}(K)$,

$$X \longrightarrow g X g^{-1}, \quad \psi^a \longrightarrow g \psi^a g^{-1}, \quad A \longrightarrow g A g^{-1} - i \partial_t g g^{-1}. \quad (3.3)$$

The supersymmetry transformation inherited from (2.5) reads

$$\delta X = i \psi^a \epsilon_a, \quad \delta \psi^a = D_t X \epsilon^a, \quad \delta A = 0. \quad (3.4)$$

Moreover, the super Yang-Mills quantum mechanics enjoys the conformal supersymmetry,

$$\delta' X = it\psi^a \epsilon'_a, \quad \delta' \psi^a = (tD_t X - X) \epsilon'^a, \quad \delta' A = 0. \quad (3.5)$$

3.2 Supersymmetric deformation and superconformal symmetry

The above super Yang-Mills quantum mechanics (3.1) admits a supersymmetric mass deformation involving an given arbitrary function of time, without breaking any of the $\mathcal{N} + \mathcal{N}$ supersymmetries, (3.4) and (3.5), as noted previously for the case of $\mathcal{N} = 1$ [41].

After introducing this given arbitrary function of time, $\Lambda(t)$, which has dimension mass squared, the deformed superconformal Yang-Mills quantum mechanics is of the general form:

$$\mathcal{L}_{\text{SYMQM}} = \text{Tr} \left[\frac{1}{2} (D_t X)^2 + i \frac{1}{2} \psi_a D_t \psi^a + \frac{1}{2} \Lambda(t) X^2 + \kappa A \right]. \quad (3.6)$$

Here we have also added a one-dimensional Chern-Simons term with coefficient (or level) κ , which must be quantized at the quantum level [42–45].

The “superconformal” symmetry or $\mathcal{N} + \mathcal{N}$ supersymmetry transformations are

$$\begin{aligned} \delta_+ X &= i f_+ \psi_a \epsilon_+^a, & \delta_+ \psi^a &= \left(f_+ D_t X - \dot{f}_+ X \right) \epsilon_+^a, & \delta_+ A &= 0, \\ \delta_- X &= i f_- \psi_a \epsilon_-^a, & \delta_- \psi^a &= \left(f_- D_t X - \dot{f}_- X \right) \epsilon_-^a, & \delta_- A &= 0. \end{aligned} \quad (3.7)$$

Here $\epsilon_+^a, \epsilon_-^b$ are $\mathcal{N} + \mathcal{N}$ supersymmetry parameters, and $f_+(t), f_-(t)$ are the two independent solutions to the second-order differential equation:

$$\ddot{f}_\pm(t) = \Lambda(t) f_\pm(t). \quad (3.8)$$

From their independence and $\frac{d}{dt} (f_+ \dot{f}_- - f_- \dot{f}_+) = 0$, it follows that $f_+ \dot{f}_- - f_- \dot{f}_+$ is a non-vanishing constant. Without loss of generality we will henceforth normalize it to unity,

$$f_+(t) \dot{f}_-(t) - f_-(t) \dot{f}_+(t) = 1. \quad (3.9)$$

In the special case of $\Lambda(t) = 0$, we may set $f_+ = 1$, $f_- = t$, and (3.7) reduces to (3.4), (3.5). On the other hand, when Λ is non-zero constant we may choose

$$\begin{aligned} f_+(t) &= \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda}t) , & f_-(t) &= \frac{1}{\sqrt{\Lambda}} \sinh(\sqrt{\Lambda}t) & \text{for } \Lambda > 0 , \\ f_+(t) &= \frac{1}{\sqrt{|\Lambda|}} \cos(\sqrt{|\Lambda|}t) , & f_-(t) &= \frac{1}{\sqrt{|\Lambda|}} \sin(\sqrt{|\Lambda|}t) & \text{for } \Lambda < 0 . \end{aligned} \quad (3.10)$$

For generic $\Lambda(t)$, under the superconformal transformations (3.7) the mass deformed Lagrangian (3.6) transforms as a total derivative,

$$\delta_{\pm} \mathcal{L}_{\text{SYMQM}} = \frac{d}{dt} \text{Tr} [D_t X \delta_{\pm} X - i \frac{1}{2} \psi^a \delta_{\pm} \psi_a] , \quad (3.11)$$

ensuring the invariance of the corresponding action.

Further, the Lagrangian (3.6) possesses a bosonic $\mathfrak{so}(\mathcal{N}) \times \mathfrak{sp}(2, \mathbb{R})$ symmetry, as follows.

- $\mathfrak{so}(\mathcal{N})$ rotation,

$$\delta_{\mathfrak{so}(\mathcal{N})} X = 0 , \quad \delta_{\mathfrak{so}(\mathcal{N})} \psi^a = M^a_b \psi^b , \quad \delta_{\mathfrak{so}(\mathcal{N})} A = 0 , \quad (3.12)$$

where $M_{ab} = -M_{ba} \in \mathfrak{so}(\mathcal{N})$.

- $\mathfrak{sp}(2, \mathbb{R}) \equiv \mathfrak{so}(1, 2) \equiv \mathfrak{sl}(2, \mathbb{R}) \equiv \mathfrak{su}(1, 1)$ conformal symmetry,

$$\delta_{\mathfrak{sp}(2, \mathbb{R})} X = \delta t D_t X - \frac{1}{2} \left(\frac{d}{dt} \delta t \right) X , \quad \delta_{\mathfrak{sp}(2, \mathbb{R})} \psi^a = 0 , \quad \delta_{\mathfrak{sp}(2, \mathbb{R})} A = 0 , \quad (3.13)$$

where δt is a generic solution to the third order differential equation [46],

$$\frac{d^3 \delta t}{dt^3} = 4\Lambda \frac{d\delta t}{dt} + 2 \frac{d\Lambda}{dt} \delta t . \quad (3.14)$$

Note that Eq.(3.13) is consistent with the fact that X has conformal weight one half. In fact, if we define

$$J_0 := i\frac{1}{2}(f_+^2 + f_-^2)\partial_t, \quad J_1 := i\frac{1}{2}(f_+^2 - f_-^2)\partial_t, \quad J_2 := if_+f_-\partial_t, \quad (3.15)$$

the three independent solutions of (3.14) can be generated by J_μ , $\mu = 0, 1, 2$.

Further, using (3.9), we obtain the commutator relations,

$$[J_0, J_1] = -iJ_2, \quad [J_1, J_2] = +iJ_0, \quad [J_2, J_0] = -iJ_1, \quad (3.16)$$

which is the Lie algebra $\mathfrak{sp}(2, \mathbb{R}) \equiv \mathfrak{so}(1, 2) \equiv \mathfrak{sl}(2, \mathbb{R}) \equiv \mathfrak{su}(1, 1)$, *i.e.* the isometry group of AdS_2 .

It is worth noting that when $\Lambda \equiv m^2$ is constant, the mass deformed SYMQM (3.6) can be identified as the radial quantization of the massless SYMQM (3.1), for which the time coordinate and the fields need to be redefined according to

$$\begin{pmatrix} t \\ X(t) \\ A(t) \\ \psi^a(t) \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{m}e^{2mt} \\ \sqrt{2}e^{mt}X(t) \\ \frac{1}{2}e^{-2mt}A(t) \\ \psi^a(t) \end{pmatrix}. \quad (3.17)$$

Otherwise, *i.e.* when $\Lambda(t)$ has nontrivial time dependence, the mass deformed SYMQM (3.6) cannot be obtained from the field redefinition of the massless SYMQM.³

³This can be traced back to the fact that the following term is not a total derivative and hence cannot be ignored for generic $\Lambda(t)$,

$$\text{Tr} \left(\sqrt{\Lambda(t)} X \frac{d}{dt} X \right).$$

3.3 Superconformal group, $\text{OSp}(\mathcal{N}|2, \mathbb{R})$

From the $\mathcal{N} + \mathcal{N}$ supersymmetry (3.7), $\text{so}(\mathcal{N})$ symmetry (3.12) and $\text{sp}(2, \mathbb{R})$ symmetry (3.13), along with (3.11) and (3.15), it is straightforward to compute the corresponding Noether charges. We write them in Hamiltonian formalism where the conjugate momentum of X is

$$P = D_t X, \quad (3.18)$$

and the Hamiltonian is independent of the fermions,

$$H = \text{Tr} \left[\frac{1}{2} P^2 - \frac{1}{2} \Lambda(t) X^2 \right]. \quad (3.19)$$

However, the equation of motion of the auxiliary gauge field A gives rise to a first-class Gauss constraint,

$$[P, X] + i\psi^a \psi_a - i\kappa = 0. \quad (3.20)$$

Upon quantization, with r, s, t, u as $K \times K$ matrix indices, we have

$$[X^r_s, P^t_u] = i\delta^r_u \delta^t_s, \quad \{\psi^{ar}_s, \psi^{bt}_u\} = \delta^{ab} \delta^r_u \delta^t_s, \quad (3.21)$$

and the left hand side of the equality in (3.20) corresponds to the $\text{U}(K)$ gauge symmetry generator.⁴ We further consider a change of the bosonic variables,

$$A_+ := f_+ P - \dot{f}_+ X, \quad A_- := f_- P - \dot{f}_- X, \quad (3.22)$$

which satisfies with the normalization (3.9),

$$[A_+^r_s, A_-^t_u] = i\delta^r_u \delta^t_s. \quad (3.23)$$

The Noether charges corresponding to all the symmetries are then as follows:

⁴This may involve a normal ordering prescription of the expression and a consequent renormalization of the Chern-Simons level.

1. Supercharges,

$$Q_+^a = \text{Tr}(\psi^a A_+) , \quad Q_-^a = \text{Tr}(\psi^a A_-) . \quad (3.24)$$

2. $\mathfrak{so}(\mathcal{N})$,

$$\mathcal{M}^{ab} = i \text{Tr}(\psi^a \psi^b) . \quad (3.25)$$

3. $\mathfrak{sp}(2, \mathbb{R}) \equiv \mathfrak{so}(1, 2) \equiv \mathfrak{sl}(2, \mathbb{R}) \equiv \mathfrak{su}(1, 1)$,

$$\mathcal{J}_0 = \frac{1}{2} \text{Tr}(A_+^2 + A_-^2) , \quad \mathcal{J}_1 = \frac{1}{2} \text{Tr}(A_+^2 - A_-^2) , \quad \mathcal{J}_2 = \frac{1}{2} \text{Tr}(A_+ A_- + A_- A_+) . \quad (3.26)$$

In order to write all the super-commutator relations in an $\mathfrak{so}(1, 2)$ covariant manner, we introduce three-dimensional 2×2 gamma matrices,

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad (3.27)$$

satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} , \quad (\gamma^\mu)^\alpha{}_\beta (\gamma_\mu)^\gamma{}_\delta = 2\delta^\alpha{}_\delta \delta^\gamma{}_\beta - \delta^\alpha{}_\beta \delta^\gamma{}_\delta , \quad (3.28)$$

where $\eta = \text{diag}(- + +)$ is the three-dimensional Minkowskian metric.

Further, we combine Q_+^a and Q_-^a to form a set of two-component Majorana spinorial supercharges, $\mathbf{Q}^{a\alpha}$, $a = 1, 2, \dots, \mathcal{N}$, $\alpha = 1, 2$,

$$\mathbf{Q}^a = \begin{pmatrix} Q_+^a \\ Q_-^a \end{pmatrix} , \quad \bar{\mathbf{Q}}_a = \mathbf{Q}^{aT} \gamma^0 = (Q_-^a , -Q_+^a) . \quad (3.29)$$

It then follows that the symmetry algebra of the superconformal Yang-Mills quantum mechanics (3.6) is the super Lie algebra $\mathfrak{osp}(\mathcal{N}|2, \mathbb{R})$ which takes the form⁵

$$\begin{aligned}
\{\mathbf{Q}^a, \bar{\mathbf{Q}}_b\} &= \delta^a_b \gamma^\mu \mathcal{J}_\mu + \mathcal{M}^a_b 1, \\
[\mathcal{J}_\mu, \mathbf{Q}^a] &= -i\gamma_\mu \mathbf{Q}^a, \\
[\mathcal{M}^{ab}, \mathbf{Q}^c] &= i(\delta^{cb} \mathbf{Q}^a - \delta^{ca} \mathbf{Q}^b), \\
[\mathcal{J}_\mu, \mathcal{J}_\nu] &= -2i\epsilon_{\mu\nu\lambda} \mathcal{J}^\lambda, \\
[\mathcal{M}_{ab}, \mathcal{M}_{cd}] &= i(\delta_{ad} \mathcal{M}_{bc} - \delta_{bd} \mathcal{M}_{ac} + \delta_{bc} \mathcal{M}_{ad} - \delta_{ac} \mathcal{M}_{bd}),
\end{aligned} \tag{3.30}$$

where $\epsilon_{\mu\nu\lambda}$ denotes the usual totally anti-symmetric tensor with $\epsilon_{012} \equiv 1$.

Finally, the Casimir of the $\mathfrak{osp}(\mathcal{N}|2, \mathbb{R})$ superalgebra is

$$\mathcal{C}_{\mathfrak{osp}(\mathcal{N}|2, \mathbb{R})} = \mathcal{J}_\mu \mathcal{J}^\mu + \frac{1}{2} \mathcal{M}_{ab} \mathcal{M}^{ab} - i \bar{\mathbf{Q}}_a \mathbf{Q}^a, \quad \left[\mathcal{C}_{\mathfrak{osp}(\mathcal{N}|2, \mathbb{R})}, \text{anything} \right] = 0. \tag{3.31}$$

⁵The Jacobi identity of the superalgebra (3.30) involving three supercharges holds due to the completeness relation of the gamma matrices in (3.28),

$$[\{\mathbf{Q}^{a\alpha}, \mathbf{Q}^{b\beta}\}, \mathbf{Q}^{c\gamma}] + [\{\mathbf{Q}^{b\beta}, \mathbf{Q}^{c\gamma}\}, \mathbf{Q}^{a\alpha}] + [\{\mathbf{Q}^{c\gamma}, \mathbf{Q}^{a\alpha}\}, \mathbf{Q}^{b\beta}] = 0.$$

4 Reduction to supersymmetric Calogero models

In this section, we discuss two possible gauge fixings which allow us to write down two types of supersymmetric generalization of the Calogero model having $\mathcal{N} + \mathcal{N}$ superconformal symmetries.

4.1 Type I: Complete gauge fixing, X diagonal and A off-diagonal

We recall the action (3.6),

$$\mathcal{L}_{\text{SYMQM}} = \text{Tr} \left[\frac{1}{2} (D_t X)^2 + i \frac{1}{2} \psi_a D_t \psi^a + \frac{1}{2} \Lambda(t) X^2 + \kappa A \right]. \quad (4.1)$$

We use the $\text{U}(K)$ gauge symmetry to fix X to be diagonal,

$$X_{ij} = 0 \quad \text{for} \quad i \neq j. \quad (4.2)$$

This still leaves a $\text{U}(1)^K$ diagonal residual gauge symmetry which can be used to eliminate the diagonal components of A ,

$$A_{ii} = 0 \quad (\text{no sum}). \quad (4.3)$$

The supersymmetry transformations (3.7) do not preserve these gauge fixings, (4.2) and (4.3), and must be compensated by an additional gauge transformation to bring us back to the gauge choice. We let

$$x_i = X_{ii} \quad (\text{no sum}), \quad (4.4)$$

and, since under a gauge transformation $\delta X = i[\lambda, X]$, we require

$$\delta_{\pm} X_{ij} = i f_{\pm} \psi_{a ij} \epsilon_{\pm}^a + i \lambda_{ij} (x_j - x_i) = 0 \quad \text{for} \quad i \neq j, \quad (4.5)$$

and in order to also preserve the off-diagonal form of A (4.3) we must choose the compensating gauge symmetry parameter as

$$\lambda_{ij} = \begin{cases} \frac{1}{x_i - x_j} f_{\pm} \psi_{a ij} \epsilon_{\pm}^a & \text{for} \quad i \neq j, \\ i \int dt [A, \lambda]_{ii} & \text{for} \quad i = j. \end{cases} \quad (4.6)$$

Note that the diagonal part of λ drops out of the commutator, $[A, \lambda]_{ii}$ in the second line. We then have the Lagrangian,

$$\mathcal{L} = \frac{1}{2}\dot{x}_i^2 + \frac{1}{2}(x_i - x_j)^2 A_{ij} A_{ji} + i\frac{1}{2}\psi_{aij}\dot{\psi}_{ji}^a + \psi_{aij}A_{jk}\psi_{ki}^a + \frac{1}{2}\Lambda(t)x_i^2, \quad (4.7)$$

where A is strictly off-diagonal (4.3) and repeated indices are summed over. Defining the combination of supersymmetry and gauge transformation (4.6)

$$\delta'_\pm = \delta_\pm + \delta_\lambda, \quad (4.8)$$

then gives $\mathcal{N} + \mathcal{N}$ supersymmetries of the Lagrangian (4.7). The original $\mathrm{U}(K)$ gauge symmetry is now completely broken.

4.2 Type II: Super Calogero model with $\mathrm{U}(1)^K$ unbroken gauge symmetry

In this subsection, we also take the diagonal gauge for X (4.2) but do not impose (4.3). Since the gauge field, A , is auxiliary we can eliminate it from the Lagrangian using its equation of motion, which is, from the Gauss constraint (3.20), given by

$$(x_i - x_j)^2 A_{ij} = \frac{1}{2}\{\psi^a, \psi_a\}_{ij} - \kappa\delta_{ij}. \quad (4.9)$$

However, the diagonal components, $a_i = A_{ii}$ (no sum), are not specified by this constraint, and we choose not to gauge them away as was done in the previous subsection. For the time being they remain as auxiliary gauge fields for the unbroken gauge group, $\mathrm{U}(1)^K$.

After eliminating the off-diagonal components of A , we have a Calogero model type Lagrangian,

$$\mathcal{L} = \frac{1}{2}\dot{x}_i^2 + i\frac{1}{2}\psi_{aij}\dot{\psi}_{ji}^a + \frac{1}{2}\Lambda(t)x_i^2 - \sum_{i \neq j} \frac{\{\psi_a, \psi^a\}_{ij}\{\psi_b, \psi^b\}_{ji}}{8(x_i - x_j)^2} + \sum_i a_i \left(\kappa - \frac{1}{2} \{\psi_a, \psi^a\}_{ii} \right), \quad (4.10)$$

which enjoys $\mathcal{N} + \mathcal{N}$ superconformal symmetries,

$$\begin{aligned} \delta x_i &= i f_{\pm} \psi_{a ii} \epsilon_{\pm}^a, \\ \delta \psi_{ij}^a &= \begin{cases} (f_{\pm} \dot{x}_i - \dot{f}_{\pm} x_i) \epsilon_{\pm}^a & \text{for } i = j, \\ i f_{\pm} \left[\sum_k \frac{\psi_{b ik} \psi_{kj}^b \epsilon_{\pm}^a}{x_i - x_j} + \sum_{k \neq i} \frac{\psi_{kj}^a \psi_{b ik} \epsilon_{\pm}^b}{x_i - x_k} - \sum_{k \neq j} \frac{\psi_{ik}^a \psi_{b kj} \epsilon_{\pm}^b}{x_k - x_j} \right] & \text{for } i \neq j, \end{cases} \quad (4.11) \\ \delta a_i &= \sum_{j \neq i} \sum_k \frac{i f_{\pm} (\psi_{a ij} \psi_{bjk} \psi_{ki}^b + \psi_{a ji} \psi_{b ik} \psi_{kj}^b) \epsilon_{\pm}^a}{(x_i - x_j)^3}. \end{aligned}$$

We see that a_i is a Lagrange multiplier for the constraint (with K components),

$$\frac{1}{2} \{\psi_a, \psi^a\}_{ii} = \kappa, \quad \text{no sum on } i. \quad (4.12)$$

If we integrate it out, then the constraint is needed to show the invariance of the action under the supersymmetries (4.11). Similarly, if we had used the $U(1)^K$ residual gauge symmetry to set the a_i to zero —as done in the previous subsection— and eliminated the off-diagonal components of A from the action (4.7) by its equation of motion, we would still need the constraint (4.12) in order to show that the action is supersymmetric. Of course imposing the constraint via a Lagrange multiplier just returns us to the action (4.10).

Unlike many known supersymmetric extensions of the multi-particle Calogero model, *e.g.* the original $\mathcal{N} = 2$ Freedman-Mende model [47], this has K^2 fermionic components compared with K bosonic (as was also the case with $\mathcal{N} = 1, 2, 4$ [9, 17]). Also, our model exists for arbitrary number of fermions and supersymmetries, and for arbitrary $\Lambda(t)$.

5 Connection to $AdS_2 \times S^{\mathcal{N}-1}$

In this section, we discuss the connection of the Abelian sector of our superconformal Yang-Mills quantum mechanics to the geodesic motion on $AdS_2 \times S^{\mathcal{N}-1}$ spacetime.

We begin by considering the standard global metric of AdS_2 with radius R ,

$$ds_{AdS_2}^2 = R^2(-\cosh^2 \rho d\tau'^2 + d\rho^2), \quad \rho \geq 0. \quad (5.1)$$

We perform a coordinate transformation following [46], from (ρ, τ') to (X, τ) by

$$\cosh^2 \rho = \frac{1}{1 - (X/R)^2}, \quad \tau = R\tau', \quad (5.2)$$

to obtain a new metric,

$$ds_{AdS_2}^2 = -\frac{d\tau^2}{1 - (X/R)^2} + \frac{dX^2}{[1 - (X/R)^2]^2}, \quad 0 \leq X < R, \quad (5.3)$$

in terms of the dimensionful variables X and τ . We use this for the metric of $AdS_2 \times S^{\mathcal{N}-1}$,

$$ds_{AdS_2 \times S^{\mathcal{N}-1}}^2 = -\frac{d\tau^2}{1 - (X/R)^2} + \frac{dX^2}{[1 - (X/R)^2]^2} + g_{\alpha\beta}(\theta) d\theta^\alpha d\theta^\beta, \quad (5.4)$$

where θ^α , $\alpha = 1, 2, \dots, \mathcal{N} - 1$ are the angular coordinates of $S^{\mathcal{N}-1}$.

After taking the temporal gauge to identify τ as the worldline coordinate, the point-particle or D0 action on $AdS_2 \times S^{\mathcal{N}-1}$ background reads

$$\mathcal{L} = -m\sqrt{[1 - (X/R)^2]^{-1} - [1 - (X/R)^2]^{-2}\dot{X}^2 - g_{\alpha\beta}(\theta)\dot{\theta}^\alpha\dot{\theta}^\beta}, \quad (5.5)$$

where m is the mass of the particle.

The corresponding Hamiltonian is

$$H = \sqrt{[1 - (X/R)^2]P_X^2 + [1 - (X/R)^2]^{-1}[m^2 + g^{\alpha\beta}(\theta)P_\alpha P_\beta]}, \quad (5.6)$$

where P_X and P_α are the canonical momenta for X and θ^α ,

$$P_X = H[1 - (X/R)^2]^{-1} \dot{X}, \quad P_\alpha = H[1 - (X/R)^2] g_{\alpha\beta}(\theta) \dot{\theta}^\beta. \quad (5.7)$$

The Hamiltonian and the Lagrangian satisfy

$$H = -\frac{m^2}{\mathcal{L}[1 - (X/R)^2]}. \quad (5.8)$$

Clearly from the form of (5.6), $g^{\alpha\beta}(\theta)P_\alpha P_\beta$ (the squared angular momentum) and the Hamiltonian itself (the energy) are conserved quantities, since they ‘commute’ with the Hamiltonian. From this, one can show straightforwardly that the Hamiltonian equation, $\dot{P}_X = -\frac{\partial}{\partial X}H$, leads to a simple harmonic oscillatory motion⁶ with frequency R^{-1} ,

$$\ddot{X} + R^{-2}X = 0. \quad (5.9)$$

Furthermore, since the $\mathfrak{so}(\mathcal{N})$ isometry of the sphere $S^{\mathcal{N}-1}$ gives rise to the Noether symmetry of the action (5.5), there are $\frac{1}{2}\mathcal{N}(\mathcal{N} - 1)$ conserved angular momenta.

Thus, the Abelian sector of the massive super Yang-Mills quantum mechanics (3.6) with the choice of $\Lambda = -R^{-2}$ describes the geodesic motion of a point-particle on $AdS_2 \times S^{\mathcal{N}-1}$, where the conserved $\frac{1}{2}\mathcal{N}(\mathcal{N} - 1)$ angular momenta are mapped to the bi-fermionic conserved quantities, $\text{Tr}(\psi_{[a})\text{Tr}(\psi_{b])}$.

⁶For a recent discussion on the connection between harmonic oscillators and AdS space, see [48].

6 Comments

As generically non-Abelian Yang-Mills quantum mechanics describes many D0-branes including their interactions, and specifically the Abelian sector of our massive super Yang-Mills quantum mechanics can be mapped to the geodesic motion of a single point-particle on $AdS_2 \times S^{\mathcal{N}-1}$, it seems natural to expect that the massive super Yang-Mills quantum mechanics (3.6) with a constant mass parameter $\Lambda = -R^{-2}$ and gauge group $U(K)$ provides a worldline description of K D0-branes on $AdS_2 \times S^{\mathcal{N}-1}$. Also, in view of the AdS_2/CFT_1 correspondence, it is desirable to investigate the non-Abelian nature of the massive super Yang-Mills quantum mechanics in order to see any ‘stringy’ features. A recent formal prescription for the computation of the correlation functions in conformal mechanics [49, 50] may help in this direction.

Also in the context of the AdS_2/CFT_1 correspondence, it would be interesting to compute the partition function of the massive super Yang-Mills quantum mechanics, $\text{Tr}(e^{-\beta H})$, with $\Lambda = -R^{-2}$. For the bosonic case of $\mathcal{N} = 0$, the quantum eigenstates are basically generated by acting with products of $\text{Tr}(\bar{C}^n)$ on the quantum vacuum, where $\bar{C} = \frac{1}{\sqrt{2}}(P\sqrt{R} + iX/\sqrt{R})$ is the matrix valued creation operator and $n = 1, 2, \dots, K$. This gives the partition function (with $q = e^{-\beta/R}$)

$$\text{Tr}(e^{-\beta H}) = q^{\frac{1}{2}K^2} \prod_{n=1}^K \frac{1}{1 - q^n}, \quad (6.1)$$

which agrees with the partition function of K bosonic harmonic oscillators [51, 52]. Further, in the large K limit or the planar limit, up to the renormalization of the overall factor, it converges to the inverse of the Dedekind eta function (which is a common special function in the computation of string theory partition functions). For the path integral derivation of the formula (6.1) see [53] and for the case of $\mathcal{N} = 2$ we refer to [54, 55]. For generic \mathcal{N} it is an open problem.

Another stringy feature of the superconformal Yang-Mills quantum mechanics, at least the massless case (3.1), is that, it can be reformulated as ‘double field Yang-Mills theory’ [56] to manifest $O(1, 1)$ T-duality. We show this in the Appendix. One more stringy or \mathcal{M} -theoretic interpretation of our model is that it may correspond to a matrix regularization of a membrane worldvolume action, where the two spatial worldvolume directions are replaced by matrix in-

dices [57, 58] (see also [59, 60] for further discussion).

In general, the universal enveloping algebras of $\mathfrak{so}(2, D - 1)$ and of the Heisenberg algebra ($[a, \bar{a}] = 1$) correspond to the D -dimensional higher spin algebra and the W_∞ algebra respectively. Since quadratic powers of Heisenberg algebra generators ($a^2, \bar{a}^2, a\bar{a} + \bar{a}a$) form the Lie algebra $\mathfrak{so}(2, 1)$, the two-dimensional higher spin algebra is a subalgebra of the W_∞ algebra. In fact, with a Z_2 -grading the W_∞ algebra can be identified as the universal enveloping algebra of the super Lie algebra $\mathfrak{osp}(1|2)$, where the Z_2 -grading distinguishes the even and odd powers in a, \bar{a} [61–67]. Our superconformal Yang-Mills quantum mechanics then generalizes the W_∞ algebra⁷ as well as the two-dimensional higher spin algebra in two ways: firstly it is supersymmetric, containing an arbitrary number, \mathcal{N} , of fermions, with the super Lie symmetry algebra, $\mathfrak{osp}(\mathcal{N}|2)$. Secondly it is non-Abelian such that there are ordering issues; *e.g.* in general, $\text{Tr}(A^2) \neq \text{Tr}(A)^2$, $\text{Tr}(A\Psi A\Psi) \neq \text{Tr}(A^2\Psi^2)$ *etc.* More precise identification of the generalized algebra is desirable.

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⁷The connection between W_∞ and abelian conformal mechanics (DFF) was previously discussed in [68, 69] as well as some superextensions in [70].

Appendix

A $\mathbf{O}(1, 1)$ T-duality covariant double field formulation

String theory possesses T-duality and imposes $\mathbf{O}(D, D)$ structure on its D -dimensional low energy effective actions [71–74]. The $\mathbf{O}(D, D)$ T-duality can be manifestly realized if we formally double the spacetime dimension, from D to $2D$, with coordinates, $x^\mu \rightarrow y^A = (\tilde{x}_\mu, x^\nu)$, and reformulate the D -dimensional effective action in terms of $2D$ -dimensional language *i.e.* tensors equipped with $\mathbf{O}(D, D)$ metric,

$$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.1})$$

This kind of reformulation was coined Double Field Theory (DFT) [75–78].

The new coordinates, \tilde{x}_μ , may be viewed as the canonical conjugates of the winding modes of closed strings [79–82]. However, in DFT, as a field theory counterpart to the level matching condition of closed string theories, it is required that all the fields as well as all of their possible products should be annihilated by the $\mathbf{O}(D, D)$ d'Alembert operator, $\partial^2 = \partial_A \partial^A$,

$$\partial^2 \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0. \quad (\text{A.2})$$

Hence locally, up to $\mathbf{O}(D, D)$ rotation, all the fields are independent of the dual coordinates [77],

$$\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0, \quad (\text{A.3})$$

and the theory is not truly doubled.

With the spacetime dimension formally doubled in double field theory, T-duality is realized by an $\mathbf{O}(D, D)$ rotation which acts on the $2D$ -dimensional vector indices of an $\mathbf{O}(D, D)$ covariant tensor in a standard manner,

$$T_{A_1 A_2 \dots A_n} \longrightarrow M_{A_1}^{B_1} M_{A_2}^{B_2} \dots M_{A_n}^{B_n} T_{B_1 B_2 \dots B_n}, \quad M \in \mathbf{O}(D, D), \quad (\text{A.4})$$

where the $\mathbf{O}(D, D)$ group is defined by the invariance of the metric (A.1),

$$M_A{}^C M_B{}^D \mathcal{J}_{CD} = \mathcal{J}_{AB}. \quad (\text{A.5})$$

While the original double field theory focused on the closed string bosonic effective actions [75–78], the understanding of the underlying differential geometry [83, 84] made it possible to construct double field Yang-Mills theory [56] as well as to include fermions [85, 86].⁸

In the current $D = 1$ case, the double field Yang-Mills theory and the $\mathbf{O}(D, D)$ covariant Dirac operators get greatly simplified due to the absence of the Kalb-Ramond field, spin connections and the Yang-Mills field strength. Further, the most general form of an $\mathbf{O}(1, 1)$ group element is given by the following simple 2×2 matrix,

$$M_A{}^B = \pm \begin{pmatrix} 0 & e^\phi \\ e^{-\phi} & 0 \end{pmatrix}. \quad (\text{A.6})$$

In terms of an einbein, e , the ‘DFT-vielbein’ [84] is given by

$$V^A = \frac{1}{\sqrt{2}} \begin{pmatrix} e \\ e^{-1} \end{pmatrix}. \quad (\text{A.7})$$

This generates a pair of projections,

$$P^{AB} = V^A V^B, \quad \bar{P}^{AB} = \mathcal{J}^{AB} - P^{AB}, \quad (\text{A.8})$$

satisfying

$$P^A{}_B P^B{}_C = P^A{}_C, \quad \bar{P}^A{}_B \bar{P}^B{}_C = \bar{P}^A{}_C, \quad P^A{}_B \bar{P}^B{}_C = 0, \quad P^A{}_B + \bar{P}^A{}_B = \delta^A{}_B. \quad (\text{A.9})$$

⁸For related yet inequivalent works, see *e.g.* [87–90].

The DFT-vielbein is in the fundamental representation of the $\mathbf{O}(1, 1)$ T-duality group, and hence from (A.6), the $\mathbf{O}(1, 1)$ T-duality simply scales the einbein.

Now, following [56], we introduce the two-component $\mathbf{O}(1, 1)$ Yang-Mills vector potential,

$$\mathcal{V}_A = \begin{pmatrix} X \\ A - e^2 X \end{pmatrix}, \quad (\text{A.10})$$

where X and A are the dynamical and auxiliary fields in the super Yang-Mills quantum mechanics (3.1).

After the above $\mathbf{O}(1, 1)$ covariant reorganization of all the field variables, using (3.17) of [56] and (4.51) of [85], the super Yang-Mills quantum mechanics (3.1) with added auxiliary einbein can be reformulated as a $D = 1$ supersymmetric double field Yang-Mills theory,

$$\mathcal{L}_{\text{SYMQM}} = \text{Tr} \left[P^{AB} \bar{P}^{CD} \mathcal{F}_{AC} \mathcal{F}_{BD} + i \frac{1}{2} \psi_a V^A \mathcal{D}_A \psi^a \right], \quad (\text{A.11})$$

which now manifests the $\mathbf{O}(1, 1)$ T-duality structure.

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